International Journal of Theoretical Physics, Vol. 3, No. 2 (1970), pp. 133–139

Gravitational and Inertial Mass of Bodies of Interacting Electrical Charges†

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Received: 3 *October* 1969

Abstract

Starting from the equations of motion of particles interacting with both electromagnetic and gravitational fields, the (passive) gravitational mass (M_g) and the inertial mass (M_i) of the total system of interacting charges is calculated. It is found that in both Einstein's General Relativity and the scalar-tensor gravitational theory of Brans and Dicke, M_g and M_i are both equal to the Special Relativistic energy of the system of interacting charged particles. Therefore, both theories are compatible with the high accuracy measurements of the M_g/M_i ratio of laboratory objects.

1. *Introduction*

In previous papers (Nordtvedt, 1968, 1969) the conditions on gravitational theories were obtained which result in massive bodies consisting of gravitationally interacting particles falling in an external gravitational field at the same rate as test bodies. This has been done by calculating the (passive) gravitational to inertial mass ration (M_g/M_i) of the massive bodies. Since these systems have been totally gravitational, gravitational theories only have been studied to obtain our results.

The experimental situation is presently as follows: there are no high-precision measurements confirming that massive gravitational systems (planets, stars) have M_g/M_i ratios which equal one. But for small laboratory objects whose internal interaction energy is electrical (and nuclear) the experiments of Eotvos (1922) and Dicke and collaborators (Roll, *et al.*, 1964) have established that M_g/M_i is the same to a part in 10^{11} for different materials.

In this paper we will incorporate electromagnetic theory into the gravitational theories, so that the gravitational and inertial mass of a collection of interacting charges can be obtained. The approach to be used will be to obtain the total equation of motion of the charges when placed in an external

† This work was supported by National Aeronautics and Space Administration grant NGR 27-001-035.

Equation (2.4) leads to a differential equation for ϕ;

$$\nabla^2 \phi = 4\pi\rho \frac{dt}{ds} g_{00} g_{ss} + \ddot{\phi} - (1+\gamma)\mathbf{g} \cdot \nabla\phi \tag{2.5}$$

where g_{ss} is the space components of $g_{\mu\nu}$;

$$g_{ss} = -(1 + 2\gamma\psi)$$

and $\mathbf{g}$ is the gravitational field;

$$\mathbf{g} = \nabla\psi$$

ρ is the invariant charge density which for a charge of strength e fulfills

$$\int \rho(\mathbf{x}) \sqrt{(-g)} \frac{dt}{ds} d^3 x = e \tag{2.6}$$

(g is the determinant of the space-time metric).

The equations of motion of the particles are obtained by making the usual variation of the action integral (2.1). The result is

$$\begin{aligned} \frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{(1-v^2)}}\right) = \nabla\psi + (\gamma + \tfrac{1}{2}) v^2 \nabla\psi - (2\gamma+1)\frac{d}{dt}(\psi\mathbf{v}) \\ + e(\nabla\phi - \dot{\mathbf{A}}) \end{aligned} \tag{2.7}$$

Needing the electrostatic field in (2.7) we solve (2.5) obtaining a solution for the electrostatic potential at $\mathbf{r}$ due to a charge e_j at $\mathbf{r}_j$;

$$\phi(\mathbf{r}) = -\frac{e_j}{|\mathbf{r}-\mathbf{r}_j|}\left(1 - \frac{1+\gamma}{2}[\psi(\mathbf{r}) + \psi(\mathbf{r}_j)]\right) + \phi_{ret}(\mathbf{r}) \tag{2.8}$$

where ϕ_{ret} is the retardation correction to the electrostatic field;

$$\phi_{ret}(\mathbf{r}) = \frac{e_j}{2}\frac{(\mathbf{r}-\mathbf{r}_j)}{|\mathbf{r}-\mathbf{r}_j|} \cdot \mathbf{a}_j + \begin{pmatrix}\text{velocity} \\ \text{dependent} \\ \text{term}\end{pmatrix} \tag{2.9}$$

with $\mathbf{a}_j$ being the acceleration of the source charge e_j. The vector potential needed in (2.7) can be used in the flat space approximation

$$\mathbf{A} = \frac{e_j \mathbf{v}_j}{|\mathbf{r}-\mathbf{r}_j|} \tag{2.10}$$

Also in (2.7) we expand

$$\frac{d}{dt}(\psi(\mathbf{r})\,\mathbf{v}) = \psi(\mathbf{r})\,\mathbf{a}_{int} + \mathbf{v}.\mathbf{g}\mathbf{v} \tag{2.11}$$

$\mathbf{a}_{int}$ is the internal acceleration of the charges produced by the interaction with the other charges.

All the terms in (2.7) can now be grouped into those proportional to the *external acceleration* of the collection of charged particles in a gravitational field and those terms proportional to the *external gravitational field.* For a collection of charged particles (m_i, e_i) in the presence of an external gravitational field (2.7) then yields

$$\left[\sum_i m_i + \tfrac{1}{2}\sum_i m_i v_i^2 + \frac{1}{2}\sum_{ij} \frac{e_i e_j}{|\mathbf{r}_i - \mathbf{r}_j|}\right]\mathbf{a}_{ex}$$

$$+\left[\sum_i m_i \mathbf{v}_i . \mathbf{a}_{ex} \mathbf{v}_i + \sum_{ij} \frac{e_i e_j}{r_{ij}^3} \mathbf{r}_{ij} . \mathbf{a}_{ex} \mathbf{r}_{ij}\right]$$

$$= \left[\sum_i m_i + \tfrac{1}{2}\sum_i m_i v_i^2 + \frac{1}{2}\sum_{ij} \frac{e_i e_j}{r_{ij}}\right]\mathbf{g} + \gamma\left[\sum_i m_i v_i^2 + \frac{1}{2}\sum_{ij} \frac{e_i e_j}{r_{ij}}\right]\mathbf{g}$$

$$-(2\gamma + 1)\left[\sum_i m_i \mathbf{v}_i . \mathbf{g}\mathbf{v}_i + \frac{1}{2}\sum_{ij} \frac{e_i e_j}{r_{ij}^3} \mathbf{r}_{ij} . \mathbf{g}\mathbf{r}_{ij}\right] \tag{2.12}$$

where we have summed the equation of motion of the particles weighted by their zeroth-order energy, m_i.

For a system of interacting charges in internal equilibrium (2.12) takes a simple and significant form. For then we can apply some virial conditions:

$$\sum_i m_i (v_i)_\alpha (v_i)_\beta + \frac{1}{2}\sum_{ij} \frac{e_i e_j}{r_{ij}^3} (r_{ij})_\alpha (r_{ij})_\beta = 0 \tag{2.13}$$

$$\sum_i m_i v_i^2 + \frac{1}{2}\sum_{ij} \frac{e_i e_j}{r_{ij}} = 0 \tag{2.13a}$$

where α and β represent any two components of the vectors. Using (2.13 and 2.13a) in (2.12) we finally get

$$\mathbf{a}_{ex} = \left(\frac{M_g}{M_I}\right)\mathbf{g} = \mathbf{g}$$

with

$$M_g = M_I = \sum_j m_j + \tfrac{1}{2}\sum m_j v_j^2 + \frac{1}{2}\sum_{ij} \frac{e_i e_j}{r_{ij}}$$

This result is independent of the particular value of the dimensionless parameter γ in the metric (2.3), as γ multiplies factors which vanish due to the virial conditions.

The inertial mass of the assembly of interacting charges is adjusted to take account of the internal kinetic and electrostatic energy contributions to the total energy of the system by means of the following properties of the equations of motion:

(1) The kinetic energy comes from the Special Relativistic energy versus velocity relationship.

(2) The electrostatic energy comes from the retardation corrections to the electrostatic field and the inductive electrical field which comes from the time rate of change of the vector potential.

The gravitational mass of the assembly of interacting charges is adjusted by the internal kinetic and electrostatic energy contributions due to the following properties of the equations of motion:

(1) The kinetic energy comes from the velocity dependence of the strength of a particle's coupling to a gravitational field.

(2) The electrostatic energy results from the curved Riemannian metric altering the Maxwell field equations and thereby the electric fields by which the charges interact with each other.

3. *Quantum Mechanical Systems*

It is evident that laboratory objects for which the M_g/M_i ratio has been measured are quantum mechanical, both at the solid state and atomic level. Equation (2.12) could be interpreted quantum mechanically by having the quantities M_g and M_i considered as operators whose expectation value is to be taken. Then using quantum mechanical analogs of the virial conditions (15 and 15a) our result becomes

$$\langle M_g\rangle = \langle M_i\rangle = \sum_j m_j + \frac{\langle(T+V)\rangle}{c^2} \tag{3.1}$$

However, this result (3.1) for the inertial mass can be obtained more convincingly by directly obtaining the Hamiltonian for interacting charges and identifying the *system's* inertial mass.

The starting point is the Breit Hamiltonian (Bethe & Salpeter, 1957) for two interacting charges (consider for simplicity identical masses). We will neglect terms in the Breit Hamiltonian which come from the intrinsic spin degrees of freedom of Dirac particles and keep only the Special Relativistic corrections to the Hamiltonian. Physically these corrections originate from the relativistic kinetic energy expression and the retardation of the Coulomb interaction.

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{p_1^4}{8m_1^3} - \frac{p_2^4}{8m_2^3} + \frac{e_1 e_2}{r_{12}} + V_{ex}(\mathbf{r}_1) + V_{ex}(\mathbf{r}_2)$$
$$+ \frac{1}{2}\frac{e_1 e_2}{r_{12}}\frac{1}{m_1 m_2}\left(\mathbf{p}_1\cdot\mathbf{p}_2 + \frac{\mathbf{r}_{12}\cdot(\mathbf{r}_{12}\cdot\mathbf{p}_1)\mathbf{p}_2}{r_{12}^2}\right) \tag{3.2}$$

$V_{ex}(\mathbf{r})$ are external potentials in which the two-particle system is placed. The change of variables are made:

$$\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M \tag{3.3}$$

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{3.3a}$$

$$\mathbf{P} = M\dot{\mathbf{R}} \tag{3.3b}$$

$$\mathbf{p} = m\dot{\mathbf{r}} \tag{3.3c}$$

with $m = m_1 m_2/M = m/2$ and $M = m_1 + m_2 = 2m$. The Hamiltonian (3.2) then reads

$$H = \frac{P^2}{2M}\left(1 - \frac{p^2}{2mM} - \frac{e_1 e_2}{Mr_{12}}\right) - \frac{1}{2M^2}\left(\frac{(\mathbf{p}.\mathbf{P})^2}{m} + \frac{e_1 e_2}{r_{12}^3}(\mathbf{r}_{12}.\mathbf{P})^2\right) \quad (3.4)$$

$$+ \frac{p^2}{2m} - \frac{p^4}{4m^3} + \frac{e_1 e_2}{2r_{12}}\left(\frac{p^2}{2mM} + 1\right) + \frac{e_1 e_2}{2r_{12}^3}\frac{(\mathbf{r}_{12}.\mathbf{p})^2}{mM}$$

Using (1.2) in a Born–Oppenheimer approximation of separation of internal and external dynamics, one can identify the expected value of the total coefficient of P^2 as $1/2M_1$ for the system. That yields

$$M_1 = 2m + \frac{\langle(T+V)\rangle}{c^2} + \frac{1}{c^2}\left\langle\left(\frac{p_{\parallel}^2}{m} + \frac{e_1 e_2}{r_{12}^3}(r_{12})_{\parallel}^2\right)\right\rangle \quad (3.5)$$

$\parallel$ refers to the components of the vectors $\mathbf{p}$ anf $\mathbf{r}_{12}$ along the direction of the external acceleration.

In a stationary quantum state the last expectation value in (3.5) vanishes and our desired result is obtained. In a mixed state the last expectation value oscillates with frequencies given by the system's energy level differences

$$\omega_{mn} = (E_m - E_n)/\hbar$$

and hence still vanishes on a time average.

In an appendix it is shown that the same result

$$M_g = M_1 = \sum_j m_j + \frac{T+V}{c^2}$$

holds for particles interacting via scalar Yukawa nuclear forces, too. The problem of treating realistic pseudo-scalar nuclear forces is complicated by the necessity of properly introducing intrinsic particle spin in a covariant way. We hope to address ourselves to that problem in a future paper.

Appendix

For particles interacting with each other via scalar Yukawa forces and also in an external gravitational field, the action integral for the Yukawa field and the particles is

$$S = \int \sqrt{(-g)}(\phi_\mu \phi^\mu - m^2\phi^2)\, d^4x/8\pi \quad (A1)$$

and

$$S = -\int (m_i + G_i\phi)\sqrt{[1 - 2\psi - v^2(1 + 2\gamma\psi)]}\, dt \quad (A2)$$

The equation of motion of the particles which results from variation of (A2) is:

$$m_i \frac{d^2\mathbf{x}_i}{dt^2} = m_i\mathbf{g} + \gamma m_i v_i^2\mathbf{g} - (2\gamma+2)m_i\mathbf{v}_i.\mathbf{g}\mathbf{v}_i$$
$$- G_i\nabla\phi(\mathbf{x}_i) + (2\gamma+2)G_i\psi(\mathbf{x}_i)\nabla\phi^{(0)}(\mathbf{x}_i) \quad (A3)$$

$\phi^{(0)}(\mathbf{x})$ is the flat-space Yukawa potential, $\psi(\mathbf{x})$ is the gravitational potential $\mathbf{g} = \nabla\psi$ is the gravitational field. The field equation for the Yukawa potential is

$$-\nabla^2\phi + (1 + 2\gamma\psi)m^2\phi - (\gamma - 1)\mathbf{g}.\nabla\phi = -4\pi G_j\rho_j(\mathbf{x})(1 - (\gamma - 1)\psi) - \ddot{\phi} \tag{A4}$$

which has the solution

$$\phi(\mathbf{x}_i) = -\sum_j G_j \frac{\exp(-mx_{ij})}{x_{ij}} \begin{Bmatrix} 1 - \gamma\psi(\mathbf{x}_j)(1 + mx_{ij}) \\ -\frac{\gamma}{2}\mathbf{g}.\mathbf{x}_{ij}(1 + mx_{ij}) \end{Bmatrix} \tag{A5}$$

Summing (A3) over all the particles and dividing by the sum of masses yields the acceleration of the system in an external gravitational field;

$$\mathbf{a} = \mathbf{g}\left[1 + \frac{1}{M}\begin{Bmatrix} \gamma\sum_i m_i v_i^2 - (2\gamma + 2)\sum_i m_i(v_i)_\parallel^2 \\ +(\gamma + 1)\sum_{ij} G_i G_j \frac{\exp(-mx_{ij})}{x_{ij}^3}(1 + mx_{ij})(x_{ij})_\parallel^2 \\ -\frac{\gamma}{2}\sum_{ij} G_i G_j(1 + mx_{ij})\frac{\exp(-mx_{ij})}{x_{ij}} \end{Bmatrix}\right] \tag{A6}$$

with

$$M = \sum_i m_i$$

which by virtue of the virial conditions for internal equilibrium of the interacting nuclear particles

$$\sum_i m_i(v_i)_\parallel^2 = \tfrac{1}{2}\sum_{ij} G_i G_j(1 + mx_{ij})\frac{\exp(-mx_{ij})}{x_{ij}^3}(x_{ij})_\parallel^2$$

$$\sum_i m_i v_i^2 = \tfrac{1}{2}\sum_{ij} G_i G_j(1 + mx_{ij})\frac{\exp(-mx_{ij})}{x_{ij}}$$

yields for arbitrary value of the parameter γ,

$$\mathbf{a} = \mathbf{g}$$

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